Spectrum formula of the synchrotron radiation from a quasiperiodic undulator

Masaru Takao* and Shigemi Sasaki
Japan Atomic Energy Research Institute–RIKEN, Spring-8 Project Team, Kamigori, Ako-gun, Hyogo 678-12, Japan

Shinya Hashimoto
Japan Synchrotron Radiation Research Institute (JASRI), Kamigori, Ako-gun, Hyogo 678-12, Japan

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Recently, two of the present authors [S. Hashimoto and S. Sasaki, Nucl. Instrum. Methods **361**, 611 (1995)] introduced the concept of a class of undulators, a quasiperiodic array of magnet poles, to discriminate the rational higher harmonics of radiation that are harmful in some synchrotron experiments. In this paper details of structural properties of radiation from a quasiperiodic undulator are developed. The analytic expression for the spectrum enables a clear understanding of the radiation from the quasiperiodic undulator with irrationally factored energies as to (i) peak positions in spectrum and (ii) peak intensities.

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I. INTRODUCTION

Ordinary undulators consist of a periodic array of magnet poles of alternating polarity. The radiation emissions in each magnet pole interfere with each other, producing enhanced emission at a fundamental frequency and its harmonics. Since a mixture of the harmonics degrades the quality of data in many experiments, the higher harmonics are required to be eliminated.

The higher harmonics of radiation are usually removed by using a total reflection mirror that reflects light below the critical energy for a given grazing angle. The other way to remove the higher harmonics is to detune a double crystal monochromator, taking advantage of the wider Darwin width of the lower harmonic. In the high-energy region of x rays around 30 keV or above, however, it is practically difficult to select a specific harmonic radiation exclusively because of a very small critical angle of the total reflection or a very narrow Darwin width.

Hashimoto and Sasaki proposed an undulator, which comprises a quasiperiodic array of magnet poles [1,2] called the "quasiperiodic undulator" (hereafter, referred to as the QPU). Since no rational harmonics but irrational ones are contained in the radiation of the QPU, the light passed through a monochromator includes no contamination of higher harmonics. That is, we are able to obtain a completely monochromatic light by the combination of a QPU and a monochromator. Using such a pure light, for example, in experiments of x-ray absorption fine structure (XAFS) and x-ray diffraction, we can detect a faint signal, which used to be buried in higher harmonic noises and also fluorescence ones.

Contrary to the conventional undulator, which generates a regular array of spectral peaks or harmonics, the QPU radiation never has such regularity but generates spectral peaks with irrationally factored energies. Any one of the spectral peaks can be selected with a conventional crystal monochromator as the purely single energy radiation without being contaminated with a mixture of harmonics.

Here we analytically formulate the QPU radiation spectrum in order to investigate what kinds of parameters are important for designing the optimum structure of quasiperiodic magnet array. We will have convenient equations to get the positions of spectral peaks and their peak intensity for a given array of magnets.

In the far field approximation the radiated intensity per electron per unit solid angle $d\Omega$ per unit frequency interval $d\omega$ is given by [3]

$$\frac{d^2I(\omega)}{d\omega d\Omega} = \frac{e^2}{16\pi^3\epsilon_0 c}\left|\int_{-\infty}^{\infty} dt\, \frac{\vec{n}\times\{[\vec{n}-\vec{\beta}(t)]\times\dot{\vec{\beta}}(t)\}}{[1-\vec{n}\cdot\vec{\beta}(t)]^2} \times e^{i\omega[t-\vec{n}\cdot\vec{r}(t)/c]}\right|^2, \qquad (1)$$

where e, ϵ_0 and c are universal constants. The vector $\vec{\beta}(t)$ is the velocity of the electron divided by c, $\vec{r}(t)$ the position, and $\vec{n}$ the unit vector oriented to the observation direction. In the case of a usual periodic undulator (PU) with N periods, this formula can be rewritten as a product of the integral over one period (form factor) and a function that reveals the interference with N successive magnetic periods (structure factor) [4–6], that is,

*On leave from JASRI, Kamigori, Ako-gun, Hyogo 678-12, Japan.

$$\frac{d^2 I(\omega)}{d\omega d\Omega} = \frac{e^2}{16\pi^3 \epsilon_0 c} \frac{\sin^2 N\pi \dfrac{\omega}{\omega_1}}{\sin^2 \pi \dfrac{\omega}{\omega_1}} \times \left| \int_0^{T_U} dt \, \frac{\vec{n} \times \{[\vec{n} - \vec{\beta}(t)] \times \dot{\vec{\beta}}(t)\}}{[1 - \vec{n} \cdot \vec{\beta}(t)]^2} \times e^{i\omega[t - \vec{n} \cdot \vec{r}(t)/c]} \right|^2 , \tag{2}$$

where ω_1 is the observed dipole frequency given by

$$\omega_1 = \frac{2\pi c}{\lambda_U (1 - \beta^* \cos\theta_0)}, \tag{3}$$

with β^* the average longitudinal velocity, θ_0 the angle between the observation direction and the undulator axis, and λ_U the period of the undulator. In Eq. (2) T_U is the time that the electron takes to traverse one undulator period λ_U. The standard analysis of Eq. (2) yields the well-known formula expressed by infinite series of Bessel functions [7–9]. The structure factor causes the radiation spectrum with a series of harmonic frequencies $\omega = \ell\omega_1$ (ℓ = integer). Since in a QPU the phase differences among the integrations over the individual magnet blocks are quasiperiodic, the structure factor gives rise to an irrational property in the spectrum. In the following section we derive an explicit expression for the radiation spectrum from a QPU.

II. SPECTRUM FOR A QUASIPERIODIC UNDULATOR

The mth quasiperiodic lattice point is represented as [1,2,10]

$$\hat{z}_m = m - (\tan\alpha - 1) + (\tan\alpha - 1) \left[\frac{\tan\alpha}{1 + \tan\alpha} m + 1 \right], \tag{4}$$

where $\tan\alpha$ is the tangent of the inclination angle of a one-dimensional (1D) quasilattice against a 2D square lattice. The symbol [ ] represents the greatest integer operator. The first term on the right-hand side in Eq. (4) corresponds to a periodic component of spacing between the lattice points, the second term represents the constant translation of lattice points, which moves the initial lattice point $\hat{z}_0$ to the origin, and as m is increased the third term quasiperiodically increases by $(\tan\alpha - 1)$ due to the irrational nature of $\tan\alpha$. Hence the distance between any two consecutive positions $(\hat{z}_m - \hat{z}_{m-1})$ takes a value of 1 or $\tan\alpha$, forming a quasiperiodic array.

A basic magnetic structure for the planer QPU can be realized by aligning positive and negative magnet poles alternately at the 1D quasilattice points designated by Eq. (4) [1,2]. From the symmetry of the 2D square lattice, where a 1D quasilattice is embedded, we can restrict $0 < \tan\alpha < 1$ without loss of generality. Thus we denote the two distances between the quasilattice points as d, $d'(= d/\tan\alpha > d)$. To realize a QPU, the length of the magnet block w should be shorter than the distance d. The alignment of the magnet poles in the case of $w = d$ is shown in Fig. 1.

As in a regular PU we assume here that the magnetic field $B_y(z)$ of the transverse QPU with N' poles has the sinusoidal dependence

$$B_y(z) = \sum_{m=0}^{N'-1} B_0 \, (-1)^m \cos\left[\frac{\pi}{w}(z - z_m)\right], \tag{5}$$

where B_0 is the peak magnetic field and z_m $(= d'\hat{z}_m)$ the center of the mth magnet region. The function $\cos(\alpha)$ is defined here to take $\cos(\alpha)$ for $-\pi/2 \le \alpha < \pi/2$ and 0 otherwise. The magnetic field distribution with $w = d$ is shown in Fig. 2.

Since the electron is accelerated only in the magnet regions, the integral in Eq. (1), being a kind of radiation amplitude, is reduced to the summation of the integrals over the magnet poles

$$\vec{A} = \sum_{m=0}^{N'-1} \int_{t_m - T_0/2}^{t_m + T_0/2} dt \, \frac{\vec{n} \times \left\{ \left[\vec{n} - \vec{\beta}(t)\right] \times \dot{\vec{\beta}}\,(t) \right\}}{\left[1 - \vec{n} \cdot \vec{\beta}(t)\right]^2} \times \exp\left[i\omega \left(t - \frac{\vec{n} \cdot \vec{r}(t)}{c} \right) \right], \tag{6}$$

where t_m is the time when the electron arrives at the center of the mth magnet region z_m and T_0 the time that the electron takes to pass through a magnet pole. Here we develop the exponent of the phase factor $i\omega\,[t - \vec{n} \cdot \vec{r}(t)/c]$. Integrating the Lorentz equation with the undulator magnetic field (5), we can write the transverse velocity as

$$\beta_x(t) = \frac{K}{\gamma} \left[\prod_{m=0}^{N'-1} \{-\mathrm{sgn}(z - z_m)\} - \sum_{m=0}^{N'-1} (-1)^m \{\sin[k_0(z - z_m)] - \mathrm{sgn}[k_0(z - z_m)]\} \right], \tag{7}$$

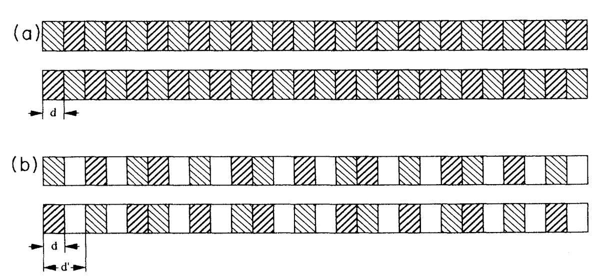

FIG. 1. Magnetic alignments of (a) a periodic undulator and (b) a quasiperiodic undulator, respectively. The hatched boxes denote magnet blocks and the blank boxes correspond to spacers.

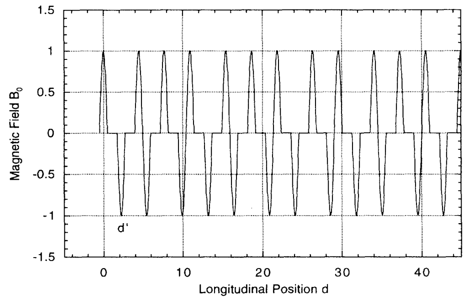

FIG. 2. Magnetic field distribution along the axis of the quasiperiodic undulator. Here B_0 denotes the peak field of the magnet block and d the length. We assume that the magnetic field in the magnet region is sinusoidal and that there is no magnetic field in spacer region.

where $\sin(\alpha)$ is the counterpart of $\cos(\alpha)$, i.e., it takes $\sin(\alpha)$ for $-\pi/2 \leq \alpha < \pi/2$ and 0 otherwise. The signature function $\mathrm{sgn}(x)$ is

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0 \end{cases} \tag{8}$$

and $\mathrm{sgn}(\alpha)$ is equal to $\mathrm{sgn}(\alpha)$ for $-\pi/2 \leq \alpha < \pi/2$ and 0 otherwise as well. For the sake of simplicity, we used the analog of the wave number $k_0 = \pi/w$ and the undulation parameter $K = (eB_0)/(m_0 c k_0)$ in the QPU. The initial transverse velocity is set to be K/γ, at which the electron runs along the undulator axis. For a highly relativistic electron, γ is very large and hence β_x is very small, so that we can approximate the longitudinal velocity $\dot{z}(t)$ as

$$\frac{\dot{z}(t)}{c} = \beta \left[1 - \frac{1}{2}\beta_x^2(t)\right]. \tag{9}$$

Then, integrating the above equation with inserting Eq. (7) into it, we obtain the longitudinal position $z(t)$ up to the order γ^{-2} as

$$\frac{z(t)}{c} = \left(1 - \frac{1 + K^2/2}{2\gamma^2}\right) t + \frac{1}{8\omega_0}\left(\frac{K}{\gamma}\right)^2 \sin\left[2\omega_0\left(t - t_m\right)\right] - \frac{1}{4}\left(\frac{K}{\gamma}\right)^2 \left(t_m - mT_0\right) \tag{10}$$

for $t_m - T_0/2 \leq t < t_m + T_0/2$. Here $\omega_0 = ck_0$ is the angular frequency of the QPU. The third term on the right-hand side represents the contribution from the free spaces between the magnet regions. Replacing the variable t by $t' + t_m$ in order to pull back the center of the magnet pole to the origin and to factorize out the phase factor, we have, for $t_m - T_0/2 \leq t < t_m + T_0/2$ or for $-T_0/2 \leq t' < T_0/2$,

$$i\omega\left(t - \frac{\vec{n}\cdot\vec{r}(t)}{c}\right) = i\frac{\omega}{\omega_1}\left[\omega_0 t' - e^{-i\pi m}\xi_x \cos\omega_0 t' - \xi_z \sin 2\omega_0 t'\right] - ik\left(\omega\right)\left(\hat{z}_m - \eta m\right), \tag{11}$$

where

$$\omega_1 = \frac{2\gamma^2\omega_0}{1 + K^2/2 + \left(\gamma\theta_0\right)^2}, \tag{12}$$

$$\xi_x = \frac{2K\gamma\theta_0\cos\phi_0}{1 + K^2/2 + \left(\gamma\theta_0\right)^2}, \tag{13}$$

$$\xi_z = \frac{K^2}{4\left[1 + K^2/2 + \left(\gamma\theta_0\right)^2\right]} \tag{14}$$

and

$$k\left(\omega\right) = \pi\frac{\omega}{\omega_2}, \tag{15}$$

$$\omega_2 = \frac{w}{d'}\frac{2\gamma^2\omega_0}{1 + K^2 + \left(\gamma\theta_0\right)^2}, \tag{16}$$

$$\eta = \frac{w}{2d'}\frac{K^2}{1 + K^2 + \left(\gamma\theta_0\right)^2}. \tag{17}$$

Here θ_0 is the polar angle of the observation direction with respect to the undulator axis and ϕ_0 the azimuthal angle measured from the undulation plane. Note that ω_1 is equal to the dipole frequency (3). The term in the first set of square brackets on the right-hand side of Eq. (11) describes the temporal motion in the magnet region and the second term represents the phase difference with respect to the zeroth magnet. In the case of a PU, the phase term becomes $i\pi(\omega/\omega_1)m$. Hence the summation of the phase factors yields a structure function containing the phase interference in Eq. (2).

Next we turn to developing the radiation amplitude (6). We carry out the integration in Eq. (6) by parts and get

$$\vec{A} = \sum_{m=0}^{N'-1}\left\{\left[\frac{\vec{n}\times[\vec{n}\times\vec{\beta}(t)]}{1 - \vec{n}\cdot\vec{\beta}(t)} e^{i\omega[t - \vec{n}\cdot\vec{r}(t)/c]}\right]_{t_m - T_0/2}^{t_m + T_0/2} - i\omega\int_{t_m - T_0/2}^{t_m + T_0/2} dt\,\vec{n}\times\left[\vec{n}\times\vec{\beta}(t)\right] e^{i\omega[t - \vec{n}\cdot\vec{r}(t)/c]}\right\} \equiv \sum_{m=0}^{N'-1}\left[\vec{A}_B(m) + \vec{A}_I(m)\right]. \tag{18}$$

In the following we denote the first term in the large curly brackets, which is called the boundary term, as $\vec{A}_B(m)$ and the second term, called the integral term, as $\vec{A}_I(m)$, respectively. Although the boundary term in the radiation amplitude of a PU vanishes due to a cancellation between the neighboring magnet poles, the boundary term in general remains finite since in the QPU the phase advances while the radiation travels through the free space. The boundary effect of the mth magnet is then written as

$$\vec{A}_B(m) = 2\gamma e^{ik(\omega)(\hat{z}_m-\eta m)}\Bigg[2i(B\gamma\theta_0\cos\phi_0 - CK, B\gamma\theta_0\sin\phi_0, 0)\sin\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right) - 2e^{-i\pi m}(C\gamma\theta_0\cos\phi_0 - BK, C\gamma\theta_0\sin\phi_0, 0)\cos\left(\frac{\pi}{2}\frac{\omega}{\omega_1}\right)\Bigg], \quad (19)$$

where

$$B = \frac{1+K^2+(\gamma\theta_0)^2}{\left[1+K^2+(\gamma\theta_0)^2\right]^2 + (2K\gamma\theta_0\cos\phi_0)^2},$$
$$C = \frac{2K\gamma\theta_0\cos\phi_0}{\left[1+K^2+(\gamma\theta_0)^2\right]^2 + (2K\gamma\theta_0\cos\phi_0)^2}.$$

We can readily see that the boundary term vanishes due to the existence of the trigonometric functions if ω is an integer multiple of ω_1 as in the case of a PU. But in the radiation amplitude of the QPU, ω is not an integer multiple of ω_1, so that the boundary term give a finite contribution to the radiation intensity. On the other hand, the mth integral term is written as

$$\vec{A}_I(m) = -i\pi\frac{\omega}{\omega_0}e^{ik(\omega)(\hat{z}_m-\eta m)} \times \sum_{\ell=-\infty}^{\infty} e^{-i\pi\ell\left(m-\frac{1}{2}\right)}\frac{\sin\left[\frac{\pi}{2}\left(\frac{\omega}{\omega_1}-\ell\right)\right]}{\frac{\pi}{2}\left(\frac{\omega}{\omega_1}-\ell\right)} \times \left(\theta_0\cos\phi_0 S_\ell^{(0)} - \frac{1}{2}\frac{K}{\gamma}\left[S_\ell^{(1)}+S_\ell^{(-1)}\right], \theta_0\sin\phi_0 S_\ell^{(0)}, 0\right), \quad (20)$$

where $S_\ell^{(p)}$ is the infinite series of Bessel functions

$$S_\ell^{(p)}\left(\frac{\omega}{\omega_1}\right) = \sum_{n=-\infty}^{\infty} J_n\left(\xi_z\frac{\omega}{\omega_1}\right)J_{2n+\ell+p}\left(\xi_x\frac{\omega}{\omega_1}\right). \quad (21)$$

Now we evaluate the summation of the amplitude over the magnet poles $\sum_{m=0}^{N'-1}\vec{A}(m)$. Factorizing out the common phase terms, we have the summation of the phase factor

$$Q_\ell(\omega) = \frac{1}{N'}\sum_{m=0}^{N'-1} e^{i[k(\omega)(\hat{z}_m-\eta m)-\pi\ell m]}. \quad (22)$$

For a PU with N periods this factor is reduced to

$$Q_\ell(\omega) = \frac{1}{2N}\sum_{m=0}^{2N-1} e^{i\pi m\left(\frac{\omega}{\omega_1}-\ell\right)} \quad (23)$$

and then

$$Q_\ell(\omega) = e^{i\pi\left(N-\frac{1}{2}\right)\left(\frac{\omega}{\omega_1}-\ell\right)}\frac{\sin\left[\pi N\left(\frac{\omega}{\omega_1}-\ell\right)\right]}{2N\sin\left[\frac{\pi}{2}\left(\frac{\omega}{\omega_1}-\ell\right)\right]}. \quad (24)$$

The insertion of the above equation into the integral (18) reproduces the radiation formula from a periodic undulator [7–9].

Since for even ℓ the second term in the exponent in Eq. (22) is an integer multiple of $2\pi i$ and then vanishes in the exponentiation, the summation $Q_\ell(\omega)$ can be simplified as

$$Q_{\ell\,\mathrm{even}}(k) = \frac{1}{N'}\sum_{m=0}^{N'-1} e^{ik(\hat{z}_m-\eta m)}, \quad (25)$$

which represents the Fourier transform of the quasiperiodic lattice except for the additional term ηm. Following the description of the quasicrystal [10], we can derive the approximated expression of $Q_\ell(\omega)$. The quasilattice point $\hat{z}_m - \eta m$ is expressed as

$$\hat{z}_m - \eta m = \left(\frac{1+\tan^2\alpha}{1+\tan\alpha} - \eta\right)m - (\tan\alpha - 1)\left\{\frac{\tan\alpha}{1+\tan\alpha}m + 1\right\}, \quad (26)$$

where the curly brackets signify the fractional part function and we have used an identity relation $x \equiv [x] + \{x\}$. It is shown [10] that Eq. (25) vanishes unless k is

$$k_{pq} = 2\pi\left(p + q\frac{\tan\alpha}{1+\tan\alpha}\right)\left(\frac{1+\tan^2\alpha}{1+\tan\alpha} - \eta\right) \quad (27)$$

with integers p and q. The exponent in Eq. (25) is then rewritten as

$$ik_{pq}(\hat{z}_m - \eta m) = 2\pi i\left(pm + q\left[\frac{\tan\alpha}{1+\tan\alpha}m + 1\right] - q\right) + iX_{pq}\left\{\frac{\tan\alpha}{1+\tan\alpha}m + 1\right\}, \quad (28)$$

where

$$X_{pq} = 2\pi q - (\tan\alpha - 1)\,k_{pq}. \quad (29)$$

The first term on the right-hand side of Eq. (28) is an integer multiple of $2\pi i$ and does not contribute to the exponentiation. Since $\tan\alpha$ is chosen to take an irrational number, we can approximate the summation over m in Eq. (25) by an integral

$$Q_{\ell\,\mathrm{even}}(k_{pq}) = \frac{1}{X_{pq}}\int_0^{X_{pq}}\exp(iy)\,dy = e^{iX_{pq}/2}\frac{\sin(X_{pq}/2)}{X_{pq}/2}. \quad (30)$$

Thus, for the large N' approximation, Eq. (25) is written as

$$Q_{\ell\,\mathrm{even}}(k) = \sum_{p,q} e^{iX_{pq}/2} \frac{\sin\left(X_{pq}/2\right)}{X_{pq}/2} \delta\left(k - k_{pq}\right). \qquad (31)$$

This implies that bright peaks occur at the discrete values of k_{pq}'s where X_{pq}'s are small and that intense peak positions (p, q) should be associated with the Fibonacci sequence [10]. Hence, in general, there appears no rational higher harmonic in the radiation spectrum from the QPU.

In the case of odd ℓ, the summation $Q_\ell(\omega)$ is reduced to

$$Q_{\ell\,\mathrm{odd}}(k) = \frac{1}{N'} \sum_{m=0}^{N'-1} e^{ik(\hat{z}_m - \eta m) - i\pi m}, \qquad (32)$$

which implies that it corresponds to the Fourier transform of the quasilattice with positive and negative matters, since the additional phase factor $\exp(-i\pi m)$ alternately changes the sign as m increasing. Then the peak position k_{pq} is shifted by the second term of Eq. (32) from the positions of ℓ even and given by

$$k_{pq} = \left[2\pi\left(p + q\frac{\tan\alpha}{1+\tan\alpha}\right) - \pi\right] \Big/ \left(\frac{1+\tan^2\alpha}{1+\tan\alpha} - \eta\right). \qquad (33)$$

In both the cases the resonant frequency of the synchrotron radiation from the QPU ω_{pq} is represented as

$$\omega_{pq} = \frac{k_{pq}}{\pi}\omega_2. \qquad (34)$$

It is emphasized that the resonant frequency of the radiation from QPU ω_{pq} has the extra K dependence through η in k_{pq} in addition to through ω_2 while the one from a PU is a simple integer multiple of ω_1.

Gathering the above results of the radiation integral, on account of the δ function in Eq. (31) we can derive the spectrum formula

$$\frac{d^2I(\omega)}{d\omega d\Omega} = \frac{e^2N'^2\gamma^2}{16\pi\epsilon_0 c} \sum_{p,q} \left(\left|\gamma\theta_0 \cos\phi_0 F_{pq} - \frac{K}{2}G_{pq}\right|^2 + \left|\gamma\theta_0 \sin\phi_0 F_{pq}\right|^2 \right) H_{pq}(\omega), \qquad (35)$$

where for an odd mode

$$F_{pq} = \left[\frac{2\tilde{k}_{pq}}{1 + K^2/2 + (\gamma\theta_0)^2} \sum_{\ell\,\mathrm{odd}} \frac{S_\ell^{(0)}(\tilde{k}_{pq})}{\frac{\pi}{2}\left(\tilde{k}_{pq} - \ell\right)} - \frac{4}{\pi}C \right] \times \cos\left(\frac{\pi}{2}\tilde{k}_{pq}\right), \qquad (36)$$

$$G_{pq} = \left[\frac{2\tilde{k}_{pq}}{1 + K^2/2 + (\gamma\theta_0)^2} \sum_{\ell\,\mathrm{odd}} \frac{S_\ell^{(1)}(\tilde{k}_{pq}) + S_\ell^{(-1)}(\tilde{k}_{pq})}{\frac{\pi}{2}\left(\tilde{k}_{pq} - \ell\right)} - \frac{8}{\pi}B \right] \cos\left(\frac{\pi}{2}\tilde{k}_{pq}\right) \qquad (37)$$

and for an even mode

$$F_{pq} = \left[\frac{2\tilde{k}_{pq}}{1 + K^2/2 + (\gamma\theta_0)^2} \sum_{\ell\,\mathrm{even}} \frac{S_\ell^{(0)}(\tilde{k}_{pq})}{\frac{\pi}{2}\left(\tilde{k}_{pq} - \ell\right)} - \frac{4}{\pi}B \right] \times \sin\left(\frac{\pi}{2}\tilde{k}_{pq}\right), \qquad (38)$$

$$G_{pq} = \left[\frac{2\tilde{k}_{pq}}{1 + K^2/2 + (\gamma\theta_0)^2} \sum_{\ell\,\mathrm{even}} \frac{S_\ell^{(1)}(\tilde{k}_{pq}) + S_\ell^{(-1)}(\tilde{k}_{pq})}{\frac{\pi}{2}\left(\tilde{k}_{pq} - \ell\right)} - \frac{8}{\pi}C \right] \sin\left(\frac{\pi}{2}\tilde{k}_{pq}\right). \qquad (39)$$

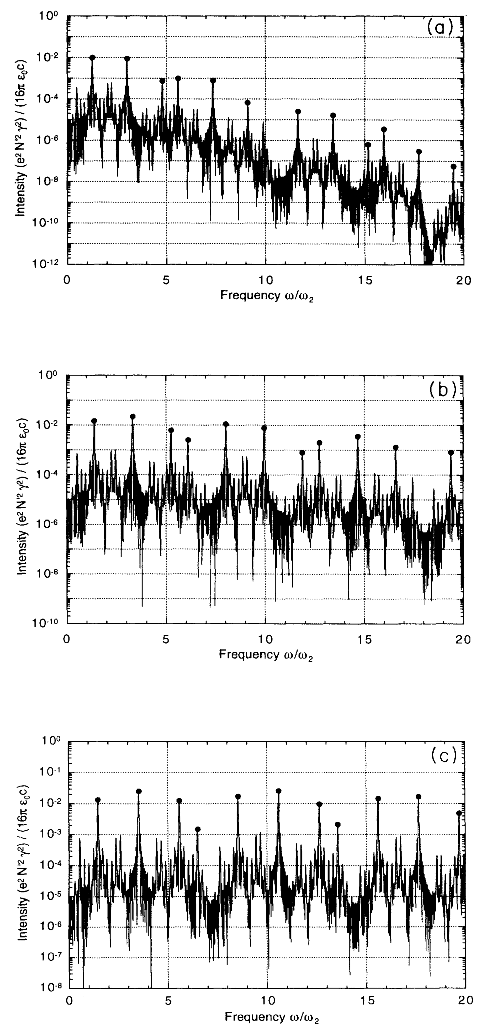

FIG. 3. Radiation spectra from the quasiperiodic undulator with K = (a) 0.5, (b) 1.0, and (c) 1.5, respectively. The solid curve indicates the numerically computed spectrum and the dots the peaks given in the analytical calculation by means of the present formula.

In the spectral formula (35) $H_{pq}(\omega/\omega_1)$ is the structural function given by

$$H_{pq}(\omega) = \left[\frac{\sin(X_{pq}/2)}{X_{pq}/2}\right]^2 \delta\left(\pi\frac{\omega}{\omega_2} - k_{pq}\right). \quad (40)$$

In Eqs. (36)–(40)

$$\tilde{k}_{pq} = \frac{d'\left[1 + K^2/2 + (\gamma\theta_0)^2\right]}{\pi w\left[1 + K^2 + (\gamma\theta_0)^2\right]} k_{pq}. \quad (41)$$

It is worth emphasizing that the intensity of the even modes vanishes on axis as seen in the case of the PU. This is because G_{pq}, consisting of $S_\ell^{(\pm 1)}$ for even ℓ and C, vanishes on axis ($\theta_0 = 0$). Furthermore, note that, although infinite Bessel series $S_\ell^{(m)}$ for some ℓ correspond to one peak in the radiation spectrum of the PU, a peak intensity of the QPU comes from all the even series or the odd series.

To confirm the validity of the formula, we compare it with the radiation spectrum numerically computed with the magnetic field given in Eq. (5) or shown in Fig. 2. Figures 3(a)–3(c) show the comparison of the radiation spectra of the QPU between the numerical and the analytical calculations with $K = 0.5$, 1.0, and 1.5, respectively. Here we take $w = d$ and $\tan\alpha = 1/\sqrt{5}$. In the numerical calculation we assume the number of poles to be 100. The full circles represent the bright peaks of the spectra designated by the generalized Fibonacci integers. Thus the analytical formula for the radiation from the QPU gives the peak position in the spectrum and the peak intensity.

III. DISCUSSION

We derived the analytical formula of the radiation from the QPU Eq. (35) under the assumption that the magnetic field is given by Eq. (5). The peak positions are given by a generalized Fibonacci sequence, indicating that no rational higher harmonic appears in general.

The assumed magnetic field in Eq. (5) may not be realistic. However, the difference from the practical magnetic field produced by a realistic magnetic array locates only in the magnet free regions where an electron orbit is straight. A small magnetic field remains in the magnet free regions in an actual magnet configuration. This difference does not cause a significant change in the spectrum formula because the main contribution to the radiation comes from the pole centers where the magnetic field possesses a large value and the acceleration of an electron is large.

The summation of the boundary terms in Eq. (18) can be interpreted as the integrals over the field free spaces as shown below. Since the velocity in the free space is constant, rearranging the boundary terms we have

$$-\left[\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{1-\vec{n}\cdot\vec{\beta}} e^{i\omega[t - \vec{n}\cdot\vec{r}(t)/c]}\right]_{t_m+T_0/2}^{t_{m+1}-T_0/2} = -i\omega\int_{t_m+T_0/2}^{t_{m+1}-T_0/2} dt\,\vec{n}\times\left(\vec{n}\times\vec{\beta}\right) \times e^{i\omega[t - \vec{n}\cdot\vec{r}(t)/c]}. \quad (42)$$

Hence, in calculating the radiation intensity from the QPU as in [2], we can use a simplified formula [3]

$$\frac{d^2 I(\omega)}{d\omega d\Omega} = \frac{e^2\omega^2}{16\pi^3\epsilon_0 c}\left|\int_{-\infty}^{\infty} dt\,\vec{n}\times[\vec{n}\times\vec{\beta}(t)] \times e^{i\omega[t - \vec{n}\cdot\vec{r}(t)/c]}\right|^2 \quad (43)$$

instead of the rather complicated formula (1). Note that in this formula the integral over the free space gives a finite contribution since the velocity $\vec{\beta}$ is finite in the field free region.

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